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Remark on cellular automata and shift preserving maps

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Abstract

The main goal of this work is to show an extension of well known Hedlund's theorem which states that in the Cantor topology the cellular automata are the continuous shift preserving maps. This extension maintains the topological structure and uses the concept of barriers of Set Theory to generalize the notion of local rule in the definition of cellular automata.

Keywords: Cellular automata; Continuous shift preserving maps; Barriers

1. Introduction

Cellular automata (CA) can be seen as discrete dynamical systems F acting on the configuration space $\mathcal{A}^{\mathbb{Z}^d} = \{x \mid x : \mathbb{Z}^d \to \mathcal{A}\}$ of all functions from $\mathbb{Z}^d$ into $\mathcal{A}$, where: $\mathcal{A}$ is a finite alphabet, $\mathbb{Z}^d$ is the d-dimensional integer lattice, and the *global transition map* F is expressed in terms of a *local rule* f which determines the evolution of each *cell* $x(n)$ $(n \in \mathbb{Z}^d)$ of the *configuration* $x \in \mathcal{A}^{\mathbb{Z}^d}$. More explicitly, let $\mathbb{V}$ be a finite and nonempty subset of $\mathbb{Z}^d$ and $\mathcal{A}^{\mathbb{V}}$ the set of all functions from $\mathbb{V}$ into $\mathcal{A}$. A local rule

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is any map $f : \mathcal{A}^{\mathbb{V}} \to \mathcal{A}$; the cellular automaton $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ given by the local rule f is defined as

$$F(x)(n) = f(x|_{\mathbb{V}+n}), \tag{1}$$

where $x|_{\mathbb{V}+n} : \mathbb{V} \to \mathcal{A}$ is given by $x|_{\mathbb{V}+n}(k) = x(n+k)$ for all $k \in \mathbb{V}$. In other words, the value of cell n in the configuration $F(x)$ is a function of the values of the cell $n + k$, with $k \in \mathbb{V}$, in the configuration x.

On the configuration space $\mathcal{A}^{\mathbb{Z}^d}$ we consider the product topology: the finest topology making continuous all the projections $\pi_n : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}$, $\pi_n(x) = x(n)$ with $n \in \mathbb{Z}^d$. With this topology $\mathcal{A}^{\mathbb{Z}^d}$ is a Cantor set (compact, perfect and totally disconnected) and every CA is a uniformly continuous map. Even if $\mathcal{A}$ is infinite but endowed with the discrete topology, the product topology of $\mathcal{A}^{\mathbb{Z}^d}$ has as a basis the class of clopen (closed and open) sets given by the collection of the *d-dimensional cylinders* $C(\mathbb{V}, h) = \{x \in \mathcal{A}^{\mathbb{Z}^d} : x|_{\mathbb{V}} = h\}$, where $\mathbb{V}$ is a finite and nonempty subset of $\mathbb{Z}^d$, h is a function from D into $\mathcal{A}$ and $x|_{\mathbb{V}}$ denotes the restriction of x to $\mathbb{V}$.

Other topological structures on $\mathcal{A}^{\mathbb{Z}^d}$ have been considered recently; two of them, Besicovitch and Weyl topologies, have shown its advantages in the study of the chaotic behavior of the CA—see for example [1] and [4]. In this work we do not consider them; nevertheless it is interesting that the problem of characterizing the continuous shift preserving maps in these topologies remains open.

There is a special class of CA. Let e_j ($j = 1, \ldots, d$) be the canonical vector in $\mathbb{Z}^d$; consider $\mathbb{V} = \{e_j\}$ and the identity map as the local rule. Then the corresponding CA is defined by $\sigma_j(x)(k) = x(k + e_j)$ for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $k \in \mathbb{Z}^d$. This d-dimensional CA is known as the shift map in the direction of e_j. An important property of the shift maps is that they commute with any other CA: $F \circ \sigma_j = \sigma_j \circ F$ for any $j \in \{1, \ldots, d\}$ for every cellular automaton F. From this property it follows that if F is a CA on $\mathcal{A}^{\mathbb{Z}^d}$, then there exists $F_0 : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}$ such that $F(x)(n) = (F_0 \circ \sigma_1^{n_1} \circ \cdots \circ \sigma_d^{n_d})(x)$, for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$.

A remarkable result known as Hedlund's theorem—see [2]—states that if $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is a continuous map commuting with σ_j for all $j \in \{1, \ldots, d\}$, then F is a CA; that is, there exists a finite set $\mathbb{V} \subset \mathbb{Z}^d$ and a function $f : \mathcal{A}^{\mathbb{V}} \to \mathcal{A}$ such that F is expressed as in Eq. (1).

This result is not satisfied if instead of a finite alphabet we consider an infinite one endowed with the discrete topology. The main interest in this work is to extend Hedlund's theorem for global transition maps acting in $\mathcal{A}^{\mathbb{Z}^d}$ and commuting with the shift maps with $\mathcal{A}$ any discrete metric space; for a proof, we will extend the notion of local rule by means of the concept of barriers of Set Theory—see [3] where a canonical form for continuous functions $\phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ that commute with the shift $S(A) = A \setminus \{\min A\}$ is obtained; there $[\mathbb{N}]^\infty$ denotes the set of all infinite subsets of $\mathbb{N}$.

2. Shift preserving continuous maps and the main result

Take an alphabet $\mathcal{A}$ endowed with the discrete topology ($\mathcal{A}$ may be infinite) and consider the configuration space $\mathcal{A}^{\mathbb{Z}^d}$ with the product topology. A map $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is *shift preserving* if it commutes with the shift maps σ_j on the configuration space $\mathcal{A}^{\mathbb{Z}^d}$. One can easily verify that F is shift preserving if, and only if, there is a map $F_0 : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}$ such that $F(x)(n) = (F_0 \circ \sigma_1^{n_1} \circ \cdots \circ \sigma_d^{n_d})(x)$, for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$. In particular, F is continuous if, and only if, F_0 is also a continuous map.

The following example shows a shift preserving continuous map F on a particular space of configuration with the product topology, which cannot be expressed by means of a local rule.

Example 1. Let $\mathbb{N}$ be the set of nonnegative integers. Define $F_0 : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}$ as $F_0(x) = \sum_{|n| \le x(0)} x(n)$. Clearly F_0 is continuous and the map $F : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ given, for every $x \in \mathbb{N}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, by

$$F(x)(n) = (F_0 \circ \sigma^n)(x) = \sum_{|j| \le \sigma^n(x)(0)} \sigma^n(x)(j)$$

commutes with the shift map σ of $\mathbb{N}^{\mathbb{Z}}$. However, F cannot be expressed in terms of a local map f as in Eq. (1).

Clearly every CA is a shift preserving continuous map, and the previous example showed us that not every shift preserving continuous map can be expressed by means of a local rule defined on uniform neighborhoods. This indicates that to characterize the shift preserving continuous maps it is necessary to modify, or to extend, the notion of local interaction rule present in the definition of CA. It is here where the notion of barrier plays a fundamental role.

Let $\mathcal{F}$ be the family of all functions $f : \mathbb{V} \to \mathcal{A}$, where $\mathbb{V}$ is any finite and nonempty subset of $\mathbb{Z}^d$. It is simple to check that $\mathcal{F}$ is partially ordered by the relation

$$f \sqsubset g \text{ if and only if } \mathrm{dom}(f) \subset \mathrm{dom}(g) \text{ and } g|_{\mathrm{dom}(f)} = f,$$

where $\mathrm{dom}(f)$ denotes the domain of f and $g|_{\mathrm{dom}(f)}$ is the restriction of g to $\mathrm{dom}(f)$.

Definition 2. An antichain of $\mathcal{F}$ is any collection in $\mathcal{F}$ of non-comparable elements with respect to $\sqsubset$. A collection $\mathcal{B} \subset \mathcal{F}$ is a barrier in $\mathcal{A}^{\mathbb{Z}^d}$ if $\mathcal{B}$ is an antichain of $\mathcal{F}$, and for every $x \in \mathcal{A}^{\mathbb{Z}^d}$ there exists a unique $f \in \mathcal{B}$ such that $x|_{\mathrm{dom}(f)} = f$. The function $f \in \mathcal{B}$ associated with x is denoted by f_x.

In view of the one-to-one correspondence between the functions $h \in \mathcal{A}^{\mathbb{V}}$ and the basic cylinders $C(\mathbb{V}, h)$, the definition of a barrier can be expressed in terms of such cylinders. Moreover, observe that for every finite and nonempty subset $\mathbb{V}$ of $\mathbb{Z}^d$, $\mathcal{A}^{\mathbb{V}}$ is a barrier of $\mathcal{A}^{\mathbb{Z}^d}$. Therefore, the local rules of CA are functions whose domains are the uniform barriers $\mathcal{A}^{\mathbb{V}}$. In this way, given a barrier $\mathcal{B}$ of $\mathcal{A}^{\mathbb{Z}^d}$, any function $\phi : \mathcal{B} \to \mathcal{A}$ will be called a *generalized local rule*.

Example 3. Let $F : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ be the shift preserving continuous map in Example 1; that is $F(x)(n) = \sum_{|j| \le \sigma^n(x)(0)} \sigma^n(x)(j)$ for every $x \in \mathbb{N}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$. We construct a barrier $\mathcal{B}$ and a generalized local rule $\phi : \mathcal{B} \to \mathbb{N}$ such that, for every $x \in \mathbb{N}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, $F(x)(n) = \phi(f_{x,n})$ where $f_{x,n} \in \mathcal{B}$ is the function associated with the configuration $\sigma^n(x)$. For each $a \in \mathbb{N} \setminus \{0\}$ and j in $\{1, 2, \ldots, a\}$ we denote by $\mathcal{B}_j^a$ the set of all function $f : [-j, j] \to \mathbb{N}$ such that $f(0) = j$ and $\sum_{|m| \le j} f(m) = a$, where $[-j, j]$ denotes the set $\{-j, \ldots, j\} \subset \mathbb{Z}$. Let $\mathcal{B} = \bigcup_{\substack{a \in \mathbb{N} \setminus \{0\} \\ 1 \le j \le a}} \mathcal{B}_j^a \cup \{f_0\}$, where $f_0 : \{0\} \to \mathbb{N}$ is given by $f_0(0) = 0$. Observe that for every pair of different functions $f : [-j, j] \to \mathbb{N}$ and $g : [-\ell, \ell] \to \mathbb{N}$ in $\mathcal{B}$, the corresponding cylinders $C([-j, j], f)$ and $C([-\ell, \ell], g)$ are disjoint; this implies that $\mathcal{B}$ is an antichain. On the other hand, given a configuration $x \in \mathbb{N}^{\mathbb{Z}}$ with $x(0) = j$, $f_x : [-j, j] \to \mathbb{N}$ where $f_x(m) = x(m)$ for every $m \in [-j, j]$ is the only function in $\mathcal{B}$ satisfying $x|_{\mathrm{dom}(f_x)} = f_x$; therefore $\mathcal{B}$ is a barrier of $\mathbb{N}^{\mathbb{Z}}$. Finally, define $\phi : \mathcal{B} \to \mathbb{N}$ by

$$\phi(f) = \begin{cases} 0, & \text{if } f = f_0 \\ \displaystyle\sum_{|m| \le j} f(m), & \text{if } f \in \mathcal{B}_j^a \text{ for some } a \in \mathbb{N} \setminus \{0\}. \end{cases}$$

Clearly for every $x \in \mathbb{N}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$ it holds that $F(x)(n) = \phi(f_{x,n})$, where $f_{x,n}$ is the function in $\mathcal{B}$ corresponding to $\sigma^n(x)$. In this way the shift preserving map F is expressed in terms of the generalized local rule ϕ.

Now we state and prove the main result of this work.

Theorem 4. *Let $\mathcal{A}$ be a discrete topological space. If $\mathcal{A}^{\mathbb{Z}^d}$ is equipped with the product topology, then:*

(a) *For every barrier $\mathcal{B}$ of $\mathcal{A}^{\mathbb{Z}^d}$ and every generalized local rule $\phi : \mathcal{B} \to \mathcal{A}$, the map $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ defined, for $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, by*

$$F(x)(n) = \phi(f_{x,n}), \qquad \text{with } (\sigma_1^{n_1} \circ \cdots \circ \sigma_d^{n_d})(x)\big|_{\mathrm{dom}(f_{x,n})} = f_{x,n} \tag{2}$$

is a shift preserving continuous map, where $f_{x,n} \in \mathcal{B}$ is the function associated with the configuration $(\sigma_1^{n_1} \circ \cdots \circ \sigma_d^{n_d})(x)$.

(b) *If $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is a shift preserving continuous map, there exists a barrier $\mathcal{B}$ of $\mathcal{A}^{\mathbb{Z}^d}$ and a generalized local rule $\phi : \mathcal{B} \to \mathcal{A}$ such that F can be expressed as in (2).*

Proof. (a) First we will show that F commutes with the shift maps of $\mathcal{A}^{\mathbb{Z}^d}$. It is clear that $(F \circ \sigma_i)(x)(n) = \phi(f_{\sigma_i(x),n})$ for $i = 1, \ldots, d$, where $f_{\sigma_i(x),n}$ is the function in $\mathcal{B}$ associated with $(\sigma_1^{n_1} \circ \cdots \circ \sigma_d^{n_d})(\sigma_i(x))$, that is,

$$(\sigma_1^{n_1} \circ \cdots \circ \sigma_d^{n_d})\ (\sigma_i(x))|_{\mathrm{dom}(f_{\sigma_i(x),n})} = f_{\sigma_i(x),n}.$$

Observe that $(\sigma_i \circ F)(x)(n) = \sigma_i(F(x))(n) = F(x)(n+e_i) = \phi(f_{x,n+e_i})$, where $f_{x,n+e_i}$ is the function in $\mathcal{B}$ such that

$$(\sigma_1^{n_1} \circ \cdots \sigma_{i-1}^{n_{i-1}} \circ \sigma_i^{n_i+1} \circ \sigma_{i+1}^{n_{i+1}} \cdots \circ \sigma_d^{n_d})\ (x)|_{\mathrm{dom}(f_{x,n+e_i})} = f_{x,n+e_i}.$$

On the other hand, since

$$(\sigma_1^{n_1} \circ \cdots \circ \sigma_d^{n_d})(\sigma_i(x)) = (\sigma_1^{n_1} \circ \cdots \sigma_{i-1}^{n_{i-1}} \circ \sigma_i^{n_i+1} \circ \sigma_{i+1}^{n_{i+1}} \cdots \circ \sigma_d^{n_d})(x),$$

it follows that $f_{\sigma_i(x),n} = f_{x,n+e_i}$ and F commutes with σ_i.

Let F_0 be the function such that $F(x)(n) = (F_0 \circ \sigma_1^{n_1} \circ \cdots \circ \sigma_d^{n_d})(x)$, for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$. To prove the continuity of F it is enough to verify the continuity of F_0. From the definition of F it follows that F_0 is given by $F_0(x) = \phi(f_x)$, where f_x is the corresponding function in $\mathcal{B}$ associated with x. As $\mathcal{A}$ is discrete, the continuity of F_0 follows from the fact that $F_0^{-1}(\{a\})$ is an open set for all $a \in \mathcal{A}$; this is clear because

$$F_0^{-1}(\{a\}) = \{x \in \mathcal{A}^{\mathbb{Z}^d} : F_0(x) = a\} = \bigcup_{x \in F_0^{-1}(\{a\})} C(f_x),$$

where $C(f_x)$ is the cylinder associated with f_x.

(b) Now consider a shift preserving continuous map F acting on $\mathcal{A}^{\mathbb{Z}^d}$; and let $F_0 : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}$ be a continuous function such that

$$F(x)(n) = (F_0 \circ \sigma_1^{n_1} \circ \cdots \circ \sigma_d^{n_d})(x),$$

for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ and every $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$. So, for any $a \in \mathcal{A}$ it follows that $F_0^{-1}(\{a\})$ is a disjoint union of basic cylinders of $\mathcal{A}^{\mathbb{Z}^d}$. Denote this collection by $\mathcal{C}_a$; and observe that $F_0(C) = a$ for all $C \in \mathcal{C}_a$. For each cylinder $C \in \mathcal{C}_a$ there exists a function $f : \mathbb{V} \to \mathcal{A}$ such that $C = C(\mathbb{V}, f)$. Let $\mathcal{B}_a$ be the collection of all these functions. It is clear that if $f, g \in \mathcal{B}_a$, then f and g are not comparable in the partial order $\sqsubset$, and for every $x \in \mathcal{A}^{\mathbb{Z}^d}$ with $F_0(x) = a$, there exists a unique function f_x in $\mathcal{B}_a$ with $x|_{\mathrm{dom}(f_x)} = f_x$. Hence $\mathcal{B} = \bigcup_{a \in \mathcal{A}} \mathcal{B}_a$ is a barrier of $\mathcal{A}^{\mathbb{Z}^d}$. Finally, if $\phi : \mathcal{B} \to \mathcal{A}$ is defined by $\phi(f) = a$ whenever the cylinder C associated with f satisfies $F_0(C) = a$, then F can be expressed as in (2) using this generalized local rule ϕ.

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